10

Waves and Optics 3

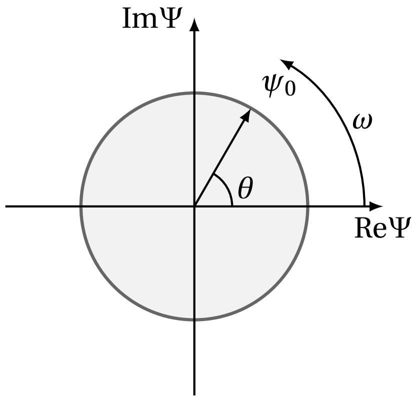
10.1 Phasors

Let's consider the wave function $\Psi(t,x) = \psi_0 \mathrm{e}^{i\theta(t,x)}$, where ψ_0 is the amplitude of the wave and $\theta(t,x) = kx - \omega t + \phi$ is the phase of the wave at a fixed position x at time t, with initial phase ϕ at $x = 0$ and $t = 0$. In the complex plane, we can represent this wave function as a vector $\Psi(t,x)$ of magnitude ψ_0 at an angle θ from the real axis. The vector $\Psi(t,x)$ precesses at an angular frequency ω around a circle of radius ψ_0, as shown in figure (10.1a). This is a *phasor*.

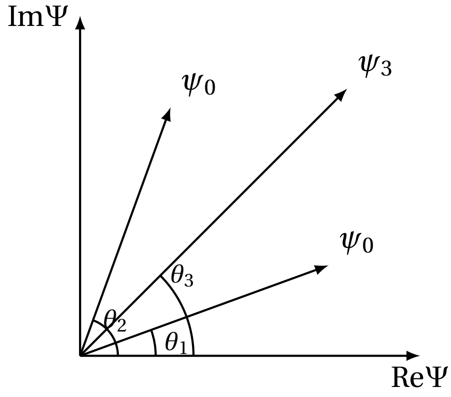
For a single wave, the phasor representation is perhaps not as useful as the algebraic equation. The power of phasors arises when we have multiple waves, perhaps with different amplitudes, as shown in figure (10.1b).

Let the phasor $\Psi_1(t,x) = \psi_0 \mathrm{e}^{i\theta_1}$ and the phasor $\Psi_2(t,x) = \psi_0 \mathrm{e}^{i\theta_2}$, i.e., the two wave functions have the same magnitude but different phases. From the diagram, we can see that Ψ_1 leads Ψ_2 by around 50 degrees. Since these wave functions satisfy the linear wave equation, we can form a linear superposition of them to form another wave function (which also satisfies the wave equation)

$$\Psi_3(t,x) = \Psi_1(t,x) + \Psi_2(t,x) = \psi_0 \mathrm{e}^{i\theta_1} + \psi_0 \mathrm{e}^{i\theta_2} = \psi_0 \left[\mathrm{e}^{i\theta_1} + \mathrm{e}^{i\theta_2} \right]. \qquad (10.1)$$



(a) The Argand diagram shows a vector $\Psi(t,x)$ of magnitude ψ_0, oriented at an angle $\theta(t,x)$ from the real axis. The vector precesses about a circle of radius ψ_0 at an angular frequency ω.



(b) Two phasors of magnitude ψ_0 and angle θ_1 and θ_2 are shown. The linear superposition of the two phasors is shown with amplitude ψ_3 and phase θ_3.

Then,

$$\begin{aligned}\Psi_3 &= \psi_0\left[e^{i\theta_1}+e^{i\theta_2}\right]\\ &= 2\psi_0 e^{i\theta_3}\cos\left(\frac{\delta}{2}\right)\\ &= \psi_3 e^{i\theta_3},\end{aligned} \tag{10.2}$$

where we have defined the new amplitude, phase and phase difference

$$\begin{aligned}\psi_3 &= 2\psi_0\cos\left(\frac{\delta}{2}\right),\\ \theta_3 &= \frac{1}{2}(\theta_1+\theta_2),\\ \delta &= \theta_2-\theta_1,\end{aligned} \tag{10.3}$$

respectively. The linear superposition continues to precess with frequency ω.

When is the phase difference δ a constant? Clearly: if the waves are coherent. If the waves were not coherent, then the phase difference and angular frequency would vary as a function of time, but we could still represent its resultant wave as a phasor—phasors work for coherent and incoherent waves.

Calculate the resultant of two coherent waves that are $2\pi/3$ out of phase.

Let us write the wave functions as

$$\Psi_1(t,x) = \psi_0 \cos(kx - \omega t), \tag{10.4}$$

and

$$\Psi_2(t,x) = \psi_0 \cos\left(kx - \omega t + \phi\right), \tag{10.5}$$

where we have identified the initial phase as $\phi = 2\pi/3$ (but we shall keep this derivation as general as possible, and substitute the value for the initial phase at the end).

Let $kx - \omega t = p$, then,

$$\Psi_3(t,x) = \Psi_1(t,x) + \Psi_2(t,x) = \psi_0\left[\cos\left(p\right) + \cos\left(p+\phi\right)\right]. \tag{10.6}$$

Using the double angle formula,

$$\begin{aligned}\Psi_3(t,x) &= \psi_0\left[\cos\left(p\right) + \cos\left(p\right)\cos\left(\phi\right) - \sin\left(p\right)\sin\left(\phi\right)\right] \\ &= \psi_0\left[\left(1+\cos\left(p\right)\right)\cos\left(p\right) - \sin\left(p\right)\sin\left(\phi\right)\right].\end{aligned} \tag{10.7}$$

Noting that $\phi = 2\pi/3$, and substituting $p = kx - \omega t$ back, one has

$$\Psi_3(t,x) = \psi_0\left[\frac{1}{2}\cos(kx-\omega t) - \frac{\sqrt{3}}{2}\sin(kx-\omega t)\right]. \tag{10.8}$$

An alternative method is to use the $A = A + B - B$ trick to write

$$\Psi_3(t,x) = \psi(kx - \omega t + \phi/2 - \phi/2) + \psi(kx - \omega t + \phi/2 + \phi/2), \tag{10.9}$$

and define $q = kx - \omega t + \phi/2$ and $\alpha = \phi/2$, such that

$$\begin{aligned}\Psi_3(t,x) &= \psi_0\left[\cos\big(q+\alpha\big)+\cos\big(q-\alpha\big)\right]\\ &= 2\psi_0\cos\big(q\big)\cos(\alpha)\\ &= 2\psi_0\cos\big(q\big)\cos\big(\phi/2\big)\\ &= \psi_0\cos\big(q\big)\\ &= \psi\cos\left(kx-\omega t+\frac{\pi}{3}\right),\end{aligned} \tag{10.10}$$

where we have noted that $\cos(\pi/3) = 1/2$, and this is equivalent to equation (10.9).