13

Electromagnetism 2

13.1 Gauss' law

We use Gauss' law to solve a **limited** set of problems in electrostatics with special symmetry: spherical, cylindrical or planar. One soon exhausts the problems that Gauss' law can be used for!

Gauss' law relates the electric field **E** on a real or imaginary surface—the Gaussian surface—to the net charge enclosed by the surface. The electric flux Φ_E is the amount of electric field that 'pierces' the Gaussian surface.

13.2 Gaussian surfaces

Gaussian surfaces are (often imaginary) constructs used to help apply Gauss' law to problems.

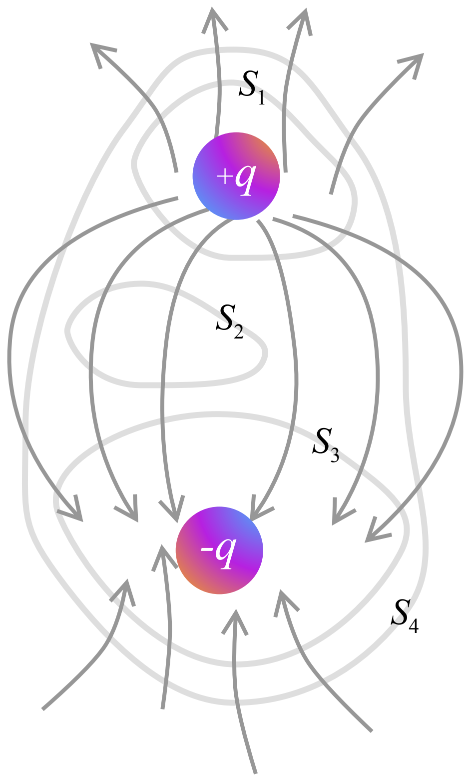

Figure 13.1: Two point charges (of opposing magnitude) are shown. Four Gaussian surfaces are constructed: surface S_1 encloses the positive charge, surface S_2 encloses an empty region of space, surface S_3 encloses the negative charge and surface S_4 encloses both charges (and S_3).

Study figure (13.1). Which statements below are true?

a) $\Phi_E = \frac{Q_{\text{enc.}}}{\varepsilon_0}$	b) Gaussian surfaces enclose all charges in the problem
c) Charges outside S_4 don't contribute to the total electric field inside S_4.	d) Charges outside S_4 don't contribute to the total electric flux through S_4.

13.3 Earnshaw's theorem

Let's consider a set of n fixed point charges $q_1, q_2, \cdots, q_n$ in a vacuum, as shown in figure (13.2). Let $\mathbf{E}(\mathbf{x})$ be the electric field produced by these charges, where $\mathbf{x}$ is an arbitrary (possibly 3-dimensional) point in the vacuum. Can we find a position $\mathbf{z}$ such that if a positive charge $q > 0$ is placed there, it will remain in a stable equilibrium?

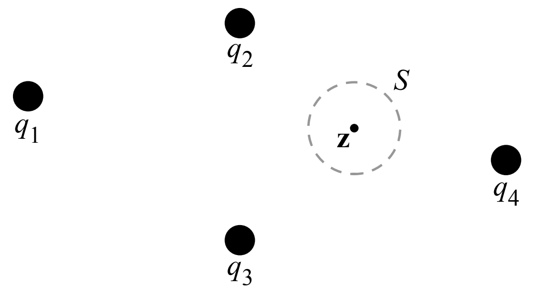

Figure 13.2: A distribution of n charges $q_1, \cdots, q_n$ are shown. A point, $\mathbf{z}$ is labelled. A Gaussian surface S encloses the point $\mathbf{z}$.

Study figure (13.2).
In which direction will an electric field $\mathbf{E}(\mathbf{x})$ point?

a) always away from $\mathbf{z}$	b) always towards $\mathbf{z}$
c) always away from $\mathbf{z}$ except at $\mathbf{x} = \mathbf{z}$	d) always towards $\mathbf{z}$ except at $\mathbf{x} = \mathbf{z}$

Divergence and the divergence theorem

[*NB: the mathematics in this section is optional.*] You will soon meet two theorems of vector calculus: the divergence theorem and Stokes' theorem. These theorems are very useful for transforming integrals involving vector fields into something easier to compute. The theorems let you move between line, surface and volume integrals for integrands involving the so-called 'divergence' or 'curl' of a vector field (which you will be introduced to soon!). We will introduce the divergence theorem now as it provides a simple mathematical description of Earnshaw's theorem.

Divergence and divergence theorem

Figure (13.3) shows examples of positive, negative and null divergence of a vector field **F**. The divergence of a vector field tells us the extent to which the field's flux behaves like a source at a given point in space. Mathematically,

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \mathbf{F}, \tag{13.1}$$

i.e., it is the dot product of a vector of first-order partial derivatives evaluated over the components of the field.

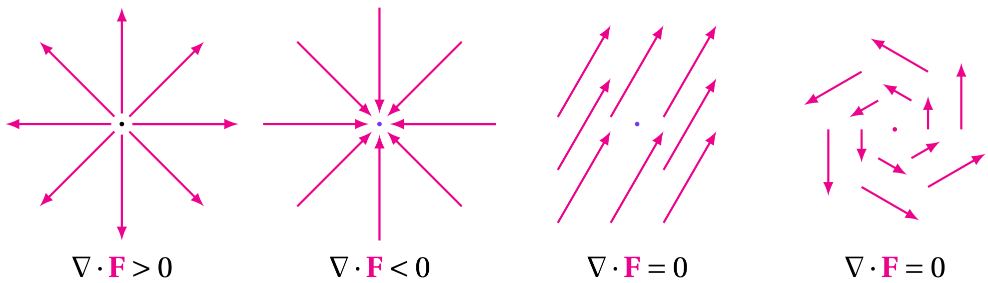

Figure 13.3: Examples of positive, negative and null divergence $\nabla \cdot \mathbf{F}$ of a vector field **F** from a central point.

Consider the vector field $\mathbf{F} = (x, y, z)$. Determine its divergence.

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x, y, z) \\ &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \\ &= 1 + 1 + 1 \\ &= 3.\end{aligned} \tag{13.2}$$

A vector field with zero divergence is said to be *solenoidal*.

Given a continuously differentiable vector field $\mathbf{F}$, the triple integral over a volume V with a piecewise smooth surface ∂V satisfies

$$\int_V (\nabla \cdot \mathbf{F})\mathrm{d}V = \oint_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{A}, \tag{13.3}$$

where $\mathrm{d}\mathbf{A}$ is an infinitesimal surface area element.

What does this mean? If we 'add up' (i.e., integrate) all the divergence in a closed volume of space V (remember, this volume is bounded by its surface, which we've called ∂V), we get the total flux coming out of that surface ∂V.

There is another important vector theorem—Stokes' theorem—which lets you move from surface integrals to line integrals. We will not cover that here.

By considering the small Gaussian surface S about z and the divergence theorem, show that a positive test charge at z cannot be in a stable equilibrium.

The test charge is in a stable equilibrium if:

- the electric force $q\mathbf{E}(\mathbf{z})$ acting on q must be zero, so $\mathbf{E}(\mathbf{z}) = \mathbf{0}$ and,
- for a small displacement $\delta\mathbf{z}$ of q about $\mathbf{z}$, there exists a restorative force $q\mathbf{E}(\mathbf{z}+\delta\mathbf{z})$ pointing back towards $\mathbf{z}$.

Since we assume that $q > 0$, $\mathbf{E}(\mathbf{z}+\delta\mathbf{z})$ must point towards $\mathbf{z}$ for the electric force on q to be restoring.

Looking at the Gaussian surface in figure (13.2), since $\mathbf{E}(\mathbf{x})$ points towards q for any point $\mathbf{x} \neq \mathbf{z}$, it must be the case that

$$\int_S \mathbf{E}\cdot \mathrm{d}\mathbf{A} < 0. \tag{13.4}$$

However, by Gauss' law,

$$\int_S \mathbf{E}\cdot \mathrm{d}\mathbf{A} = \int_V \nabla\cdot\mathbf{E}\,\mathrm{d}V = \frac{Q_{\text{enc.}}}{\varepsilon_0} < 0. \tag{13.5}$$

This goes against our initial assumption that $Q_{\text{enc.}} = 0$ (as q itself does not contribute to the electric field $\mathbf{E}(\mathbf{z})$. There is therefore no point of stable equilibrium in an electrostatic field.

Earnshaw's theorem (1842) has consequences for atomic stability and gives rise to quantum mechanical descriptions of atomic stability.