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Electromagnetism 3

14.1 Infinite, charged sheet

Let us consider the infinite plane sheet with a uniform charge per unit area σ, as shown in figure (14.1).

From Gauss' law, we know that

$$\Phi_E = \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q_{\text{enc.}}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}. \tag{14.1}$$

Since the electric field only acts perpendicular to the surface, there is no flux coming through the sides of the cylinder (i.e., parallel to the infinite sheet). The only contributions to the electric flux are through the top and bottom of the cylindrical Gaussian surface, i.e., $\Phi_E = 2AE$. Therefore,

$$E = \frac{\sigma}{2\varepsilon_0}. \tag{14.2}$$

An infinite sheet of constant charge creates a constant electric field.

The magnitude of the electric field is independent of the distance from the sheet d. We can show this by dimensional analysis. The only length scale in the problem is d, since the sheet is infinite. The electric field is therefore some function of ε_0 (the electrical permittivity is dependent upon), σ and d,

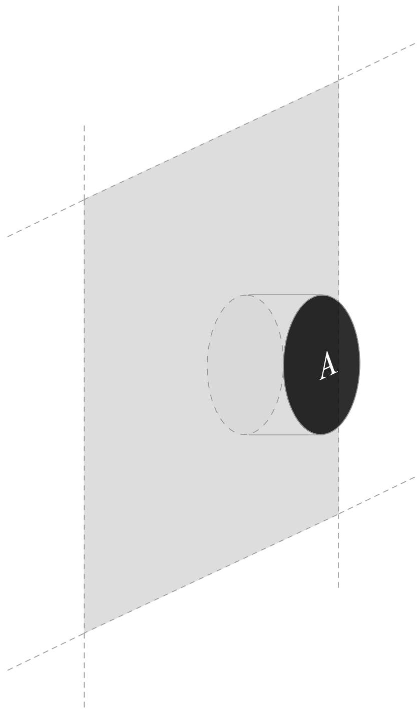

Figure 14.1: A segment of an infinitely wide sheet is shown. A cylindrical Gaussian surface of cross-sectional area A is also shown.

i.e., $E \equiv E(\varepsilon_0, \sigma, d)$. Performing dimensional analysis,

$$\begin{aligned} E &\propto \varepsilon_0^\alpha \sigma^\beta d^\gamma \\ \implies [E] &= [\varepsilon_0^\alpha \sigma^\beta d^\gamma] \\ \implies MLT^{-2}(IT)^{-1} &= [(IT)^2(MLT^{-2})^{-2}L^{-2}]^\alpha [L^{-2}IT]^\beta L^\gamma \\ \implies (\alpha, \beta, \gamma) &= (-1, 1, 0). \end{aligned} \tag{14.3}$$

There is no dependence on the electric field on the length scale d since $\gamma = 0$.

Find an equation for the electrical capacitance of a charged sphere by i) dimensional analysis and ii) through the definiton of capacitance.

i) The capacitance only depends on the geometry of the sphere (i.e., its radius R). Other length scales are introduced through the permittivity of free space ε_0. Thus,

$$\begin{aligned} C &\propto \varepsilon_0^\alpha R^\beta \\ \implies [C] &= [\varepsilon_0^\alpha R^\beta] \\ \implies (IT)(ML^2)^{-1} &= [(IT)^2(MLT^{-2})^{-2}L^{-2}]^\alpha L^\beta \\ \implies (\alpha,\beta) &= (1,1) \end{aligned} \tag{14.4}$$

So the equation for capacitance is

$$C(R) = c_0\varepsilon_0 R, \tag{14.5}$$

where c_0 is a constant of proportionality.

ii) We can also build a model using equations that we know. Let us assume first that the charged sphere (of radius a) is made of a charged surface, a gap, and then a smaller charged sphere (of radius b) inside of it (to form a capacitor). We know that the electric field outside of a charged, conducting sphere is

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \tag{14.6}$$

The voltage between the spheres is

$$\Delta V = \int_a^b \frac{Q}{4\pi\varepsilon_0 r^2}\mathrm{d}r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]. \tag{14.7}$$

From the definition of capacitance,

$$C = \frac{Q}{\Delta V} = 4\pi\varepsilon_0\left[\frac{1}{a} - \frac{1}{b}\right]^{-1}. \tag{14.8}$$

Taking the smaller radius $a \to R$ and the larger radius to infinity $b \to \infty$, one finds that

$$C = 4\pi\varepsilon_0 R. \tag{14.9}$$

We can identify $c_0 = 4\pi$ from the dimensional analysis.