15

Quantum Mechanics 1

15.1 Photoelectric effect

The photoelectric effect is the emission of electrons when photons are incident upon the surface of a material. Each photon carries energy $E = h\nu$, i.e., the photon energy is directly proportional to the frequency ν on the electromagnetic spectrum. Photons can only ionise an atom on the surface of a material if its energy exceeds the binding energy of the electron. The maximum kinetic energy of such electrons is

$$T_{\text{max}} = h\nu - \Phi, \tag{15.1}$$

where Φ is the work function of the material. The ejected electrons are sometimes called photoelectrons. When $h\nu < \Phi$, $T_{\text{max}} < 0$ and so no photoelectrons are observed.

The work function is the minimum energy needed to remove an electron from a solid to a point in the vacuum immediately outside the solid. This is actually quite far on the atomic scale, but not close enough for the electron to be influenced by any electric fields in the vacuum. The work function is a surface effect, and is entirely independent of the bulk of the material.

Which statements are true?

a) $T_{\max}$ increases if the light intensity increases	b) $T_{\max}$ increases if the light frequency increases
c) If the light intensity is halved, the number of photoelectrons produced is also halved	d) If the light intensity is halved, the photoelectron speed reduced by $\sqrt{2}$

15.2 de Broglie wavelength

The de Broglie wavelength λ of a massive particle is related to its (relativistic) momentum via

$$\lambda = \frac{h}{p} = \frac{h}{\gamma m_0 v}, \tag{15.2}$$

where γ is the Lorentz factor, v is the velocity of the particle and m_0 is the rest mass of the particle.

Let us assume that $v \ll c$ such that γ is of the order of unity. Then,

$$\lambda \approx \frac{h}{m_0 v}. \tag{15.3}$$

It is always more convenient to work in units of eV/c^2 in particle physics problems (this just comes from $E = mc^2$). It is a crime against physics if you convert such masses to kg. There are too many ways to make mistakes if you convert everything to metres, kilograms and seconds and back again. Rounding issues are inevitable when dealing with powers of c. How do we use units of eV/c^2 in our formula for the de Broglie wavelength?

We first recognise that we have our ratio $\beta = v/c = 10^{-2}$ (this is a slight restatement of the speed in the question). We can introduce this ratio into equation

(15.3) such that

$$\lambda = \frac{hc}{m_0 c^2 \left(\frac{v}{c}\right)}, \tag{15.4}$$

where we have balanced factors of c on the top and bottom. We have also incidentally introduced the factor of c^2, which will become useful.

Throughout the tutorial course, I have always said to re-arrange first and then substitute in numbers. Please prepare yourself as I now break this rule. I recognise that factors of eV and c^2 appear in equation (15.4), but so far I am unsure how many. Let us check by substituting in $\hbar = 6.6 \times 10^{-19}$ eV s and $c = 3.0 \times 10^8$ m s^{-1}

$$\lambda = \frac{2\pi \times 6.6 \times 10^{-19}\ [\mathrm{eV\,s}] \times 3.0 \times 10^8\ [\mathrm{m\,s^{-1}}]}{m_0 c^2 \left(\frac{v}{c}\right)}. \tag{15.5}$$

The constant on the numerator can be evaluated as $hc = 1.2 \times 10^{-6}$ eV m. I am now going to perform some mathematical parkour by moving the unit of eV to the denominator such that

$$\lambda = \frac{1.2 \times 10^{-6}\ [\mathrm{m}]}{\left(\frac{m_0 c^2}{\mathrm{eV}}\right)\left(\frac{v}{c}\right)}. \tag{15.6}$$

We have now conveniently rearranged our equation to explicitly include masses in units of eV/c^2. Let us use this equation in an example.

Estimate the de Broglie wavelength for (a) a proton and (b) an electron, when both are moving with speed $10^{-2}c$.

[*The following values may be helpful:* $m_p = 1\ \mathrm{GeV/c^2}$, $m_e = m_p/10^3$]

We determine the de Broglie wavelength of the particles using equation (15.6).

Given that $m_p = 1\frac{\mathrm{GeV}}{c^2}$, this implies that

$$\frac{m_p c^2}{\mathrm{eV}} = 10^9, \tag{15.7}$$

since $1\ \mathrm{GeV} = 10^9\ \mathrm{eV}$. Then,

$$\lambda_p = \frac{1.2 \times 10^{-6}\ [\mathrm{m}]}{10^9 \cdot 10^{-2}} = 1.2 \times 10^{-7}\ \mu\mathrm{m}. \tag{15.8}$$

We can quickly note that $\lambda_e = 10^3 \lambda_p = 1.2 \times 10^{-4}\ \mu\mathrm{m}$.

[Aside: we can further note that $\lambda_e \sim 10^{-10}\ \mathrm{m} = 1\ \text{Å}$. This is comparable to the Bohr radius $a_0 \sim 10^{-11}$ m—the mean orbital radius of an electron around the nucleus of a hydrogen atom in its ground state. In the Bohr model, $v/c = 10^{-2}$, as we have used. The Bohr radius is a constant widely used in atomic and molecular quantum mechanics.]

15.3 Blackbody radiation

A blackbody is a surface which absorbs all radiant energy (energy from the electromagnetic spectrum) incident upon it. Wien's displacement law tells us that the spectral radiance of a blackbody per unit wavelength peaks at the wavelength λ_{max}, given by

$$\lambda_{\mathrm{max}} = \frac{a}{T}, \tag{15.9}$$

where a is a constant and T is the absolute temperature. Whilst not particularly important, the *spectral* radiance is the radiance of a surface per unit wavelength (or per unit frequency). It should not be confused with spectral intensity, which is not a surface effect. Spectral radiance as a function of wavelength is commonly given in units of $\mathrm{W\,sr^{-1}\,m^{-2}\,nm^{-1}}$, i.e., watts per steradian (solid angle) per unit area per unit wavelength.

Wien's displacement law is actually an entirely quantum mechanical effect. Let us look at two other descriptions of the spectral radiance: Planck's law and the Rayleigh–Jeans law. Planck's law and the Rayleigh–Jeans law were developed independently in the early 1900s. The Rayleigh–Jeans law says that the spectral radiance is

$$B_\lambda(T) = \frac{2ck_\mathrm{B}T}{\lambda^4}, \tag{15.10}$$

where c is the speed of light and k_B is the Boltzmann consant. In the limiting case $\lambda \to 0$, we see that $B_\lambda \to \infty$. This implies that as the wavelength approaches the ultraviolet part of the electromagnetic spectrum, the total energy output diverges to infinity. Experimental evidence disagreed with this—it was observed that the spectral emission reaches a maximum and falls as wavelength decreases, so the total energy output is finite. This divergence of B_λ is known as the *ultraviolet catastrophe*. The failure arises in the purely classical physics derivation of the Rayleigh–Jeans law—it does not consider quantum mechanics.

Planck's law states that

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda k_\mathrm{B}T) - 1}, \tag{15.11}$$

which avoids the ultraviolet catastrophe as $\lambda \to 0$. Planck's law is quantum mechanical as it considers the quantisation of radiation. It is this law which led to the development of modern quantum theory.

Wien's approximation of the spectral radiance is

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\exp\left(-\frac{hc}{\lambda k_\mathrm{B}T}\right), \tag{15.12}$$

which is very similar to Planck's law. Wien developed this law many years before Planck, and was derived before Planck's introduction of the quantisation of radiation.

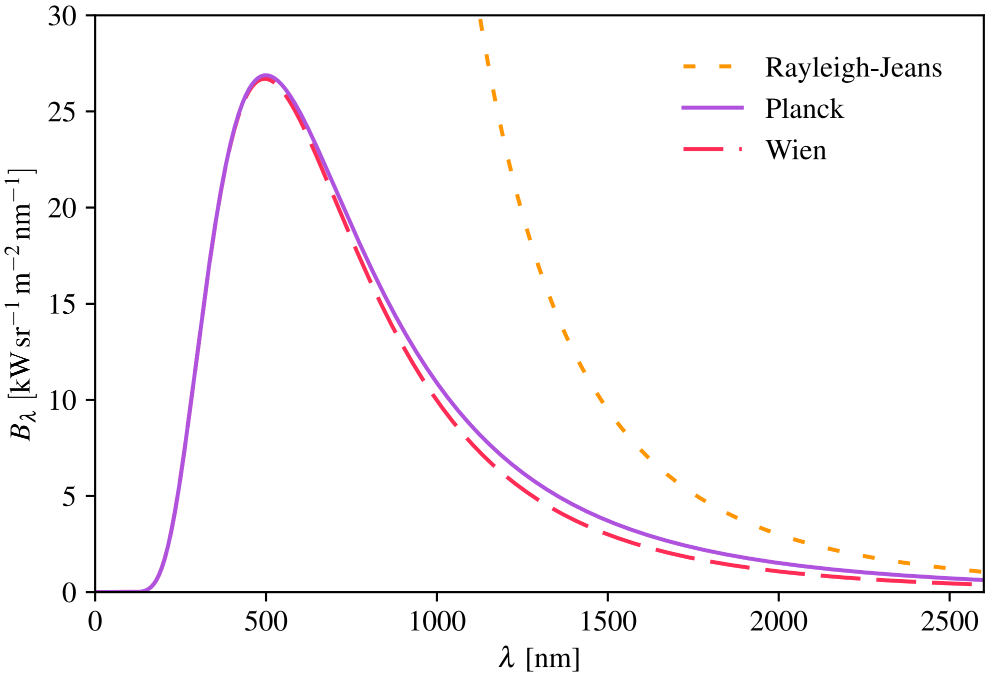

Figure 15.1: A comparison of the spectral radiance, B_λ is shown as a function of wavelength, $\lambda \in [0, 2500]$nm, as predicted by the Rayleigh-Jeans law (short dashes - -, orange), Planck's law (long dashes – –, pink) and Wien's law (solid line —, purple).

Wien's law and Planck's law agree in the ultraviolet range. Planck's law is the most accurate description of spectral radiation.

The spectrum of the Sun is approximately a blackbody with surface temperature $T_{\odot} = 5777$ K. Consider a star which is the same radius as the Sun, but with a surface temperature $T = 2T_{\odot}$.

Which of the following are true?

a) The total power radiated would be twice that of the Sun	b) The colour of the star would be redder than the Sun
c) $\lambda_{\max}$ would halve	d) The rate at which nuclear fuel is burnt would double