15

Relativity 2

15.1 Lorentz transformations

Let us consider two frames of reference, S and S' [1]. Let the constant relative speed of the frames be v, and let the axes of each frame point in the same direction. Finally, let the origin of the S' frame move along the x-axis of the S frame. In 1D, there is no relativistic motion in the y and z axes.

Lorentz transforms do not 'transform' coordinates from one frame to another. Strictly speaking, they tell you the differences Δx and Δt of the coordinates in one frame to the $\Delta x'$ and $\Delta t'$ in another frame. We are interested in the difference of spacetime coordinates between two events. The Lorentz transformations are

$$\begin{aligned} \Delta x &= \gamma(\Delta x' + v\Delta t') \\ \Delta y &= \Delta y' \\ \Delta z &= \Delta z' \\ \Delta t &= \gamma(\Delta t' + v\Delta x'/c^2). \end{aligned} \tag{15.1}$$

[1] There is nothing special about the primed frame and the unprimed frame. One frame is no more important than the other. It does not make sense to ask which frame is moving. In fact, I dislike this notation a lot. You should use meaningful subscripts depending on the problem at hand (e.g., 'B' for barn, 'P' for pole).

Lorentz transformations imply the loss of simultaneity, time dilation and length contraction

The Lorentz transformations are magical. They imply the loss of simultaneity, time dilation and length contraction:

- **Loss of simultaneity:** let two events occur simultaneously in a frame S', such that $(\Delta x', \Delta t') = (\Delta x', 0)$. The Lorentz transformation of Δt implies that $\Delta t = \gamma \Delta x'/c^2$, which is not equal to zero unless $\Delta x' = 0$, so the events do not occur simultaneously in S.
- **Time dilation:** let two events occur in the same place in S', such that $(\Delta x', \Delta t') = (0, \Delta t')$. The Lorentz transformation of Δt implies that

  $$\Delta t = \gamma \Delta t' \tag{15.2}$$

  Since $\gamma \geq 1$, $\Delta t > \Delta t'$, so moving clocks tick slower. This also works for the inverse transformations.
- **Length contraction:** let a stick with proper length $\ell_\star$ be at rest in S'. Let length of the stick in the frame S is ℓ. We measure the difference in the positions of the end of the stick in S such that $(\Delta x, \Delta t) = (\Delta x, 0)$. The Lorentz transformations imply that $\Delta x' = \gamma \Delta x$. By definition, $\Delta x'$ is the proper length $\ell_\star$ and Δx is the length in S, thus

  $$\ell = \ell_\star / \gamma, \tag{15.3}$$

  so moving rods appear shorter.

What happens if we interchange the two frames S and S' (remember, no frame is more important than the other)? For the case of time dilation, we enforced that $\Delta x' = 0$. We could switch the frames S and S' and consider the separation $(\Delta x, \Delta t) = (0, \Delta t)$. We need to use the inverse Lorentz transformations, or simply note that $\Delta t \to \Delta t'$ and $v\Delta x' \to -v\Delta x$, such that

$$\Delta t' = \gamma(\Delta t - v\Delta x/c^2). \tag{15.4}$$

When $\Delta x = 0$, we obtain

$$\Delta t' = \gamma \Delta t. \tag{15.5}$$

Compare equation (15.2) and equation (15.5). Is there a contradiction? The answer is no! We have based these equations on two totally different assumptions: equation (15.2) assumes that $\Delta x' = 0$ and equation (15.5) assumes that $\Delta x = 0$, and $\Delta x' \neq \Delta x$. The two equations have nothing to do with each other, and you must be concious of this. The same argument applies to length contraction (this is left as an exercise for the reader.)

Imagine that a distant civilisation on the opposite side of the galaxy, 100,000 light years away from us, launched their first rocket 50,000 years before this was done on Earth (as measured in our frame of reference). A spacecraft is travelling from that distant point of the galaxy towards Earth at a speed of 0.99c, following a straight line trajectory. What interval of time would separate these two events in the frame of the spacecraft?

Let P_1 be the event of the distant civilisation launching their rocket and P_2 be the event of the rocket launch on Earth.

Let $\Delta x = 100,000$ ly and $\Delta t = 50,000$ years. With $v = 0.99c$, $\gamma = 7.09$.

Thus,

$$\Delta t' = 7.09 \times (50,000 - 0.99 \times 100,000) = -350,000 \text{ years.} \tag{15.6}$$

In other words, in the frame of the spacecraft, the rocket was launched on Earth about 350,000 years before one was launched by the distant civilisation.

The rocket is accelerated by a constant force (which points in the direction of motion) from rest. How much kinetic energy does the rocket gain over 1 meter of travel when it is at 0.99 c, compared to the kinetic energy it has gained over the first meter of travel?

In the lab frame the force is constant and the distance is the same in the two cases, therefore the kinetic energy gained is the same.

The acceleration is γ^3 smaller, since $F = \gamma^3 ma$.

Spacetime axis contractions

The factor of ux/c^2 is not only dimensionally correct (it has dimensions of time), but corresponds to the lack of synchronisation between clocks in two different reference frames. Figure (15.1) shows two sets of axes—the primed axes are moving at a constant velocity away from the unprimed axes. The effect of moving at a constant, relativistic velocity is that the axes are transformed and 'squished' towards each other—this is the Lorentz transformation in action.

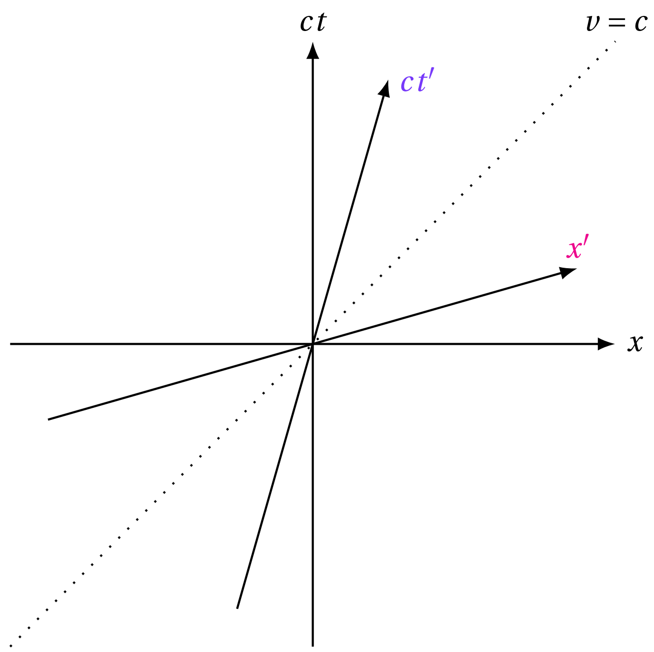

Figure 15.1: A spacetime diagram shows ct as a function of x for the unprimed frame. Different axes are shown for the primed frame—these axes are transformed by some amount compared to the unprimed frame. The line $v = c$ is shown for reference. The primed axes contract following hyperbolic curves.

15.2 Relativistic dynamics

In relativistic mechanics, when you double the speed of a particle, its momentum increases by

a) a factor of 2	b) a factor greater than 2
c) a factor between 1 and 2	d) a factor less than 1

The relavistic energy of the rest mass m_0 is

$$E = \gamma m_0 c^2, \tag{15.7}$$

and the relativistic linear momentum is

$$p = \gamma m v^2. \tag{15.8}$$

By the *correspondance principle*, the non-relativistic limits of equation (15.7) and equation (15.8) can be obtained by a Taylor expansion of γ. For low $\beta = v/c$, and to first order in v^2/c^2,

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}. \tag{15.9}$$

Thus the relativistic energy is

$$E \approx \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) m_0 c^2 = m_0 c^2 + \frac{1}{2} m v^2, \tag{15.10}$$

i.e., the energy of the rest mass plus its classical kinetic energy. Similarly, for momentum,

$$p \approx \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) m v \approx m v. \tag{15.11}$$

The energy-momentum relation holds for relativistic bodies

$$\begin{aligned} E^2 - p^2c^2 &= \gamma^2 m^2 c^4 - \gamma^2 m^2 v^2 c^2 \\ &= \gamma^2 m^2 c^4 \left(1 - \frac{v^2}{c^2}\right) \\ &= m^2 c^4, \end{aligned} \tag{15.12}$$

where the factors of γ^2 cancel.

For massless objects (e.g., photons), $E = pc$. Equation (15.7) and equation (15.8) are undetermined for 0 mass and infinite Lorentz factors, so it makes no sense to use them.

Given p and E, the quickest way to find v is to combine equation (15.7) and equation (15.8) to find

$$\frac{p}{E} = \frac{v}{c^2}. \tag{15.13}$$

The total energy is $E = \gamma mc^2$. The kinetic energy is a particle's excess energy over the energy it has when it is motionless, i.e.,

$$T = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2. \tag{15.14}$$

Using the binomial expansion, show that for a same mass and a same speed the relativistic kinetic energy is always greater than the non-relativistic kinetic energy.

Performing a binomial expansion on $\gamma - 1$ gives

$$\gamma - 1 = 1 + \left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right) - 1 \approx \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4}. \tag{15.15}$$

Classically, $T_{\text{cla.}} = \frac{1}{2}mv^2$. Relativistically, using this binomial expansion,

$$T_{\text{rel.}} \approx \frac{1}{2}\frac{v^2}{c^2}mc^2 + \frac{3}{8}\frac{v^4}{c^4}mc^2 = \frac{1}{2}mv^2 + \frac{3}{8c^2}mv^4 > \frac{1}{2}mv^2, \tag{15.16}$$

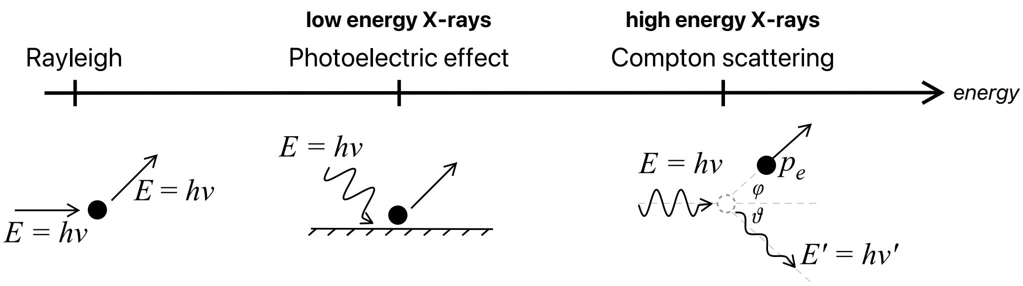

Figure 15.2: Three scattering effects are shown, in order of the energy of the X-ray radiation used to trigger those effects. Rayleigh scattering is a low energy effect. The photoelectric effect is a higher energy effect. Compton scattering is an even higher energy effect. Schematics showing the interaction of light with matter are shown. Electrons are shown as solid black circles. In the Compton scattering schematic, the electron before its collision with a photon is shown in a dashed grey circle.

so the relativistic kinetic energy is always greater than the classical kinetic energy.

Example of relativistic dynamics: Compton scattering

Compton scattering is an effect of both quantum mechanics and special relativity. When light interacts with matter, one of a variety of effects may be observed: Rayleigh scattering, the photoelectric effect, Compton scattering or pair production. Rayleigh scattering is a low energy effect, whilst pair production is a very high energy effect. The probability of one of these scattering events occuring varies depending upon the energy of the incident radiation used (you will meet this in stage 3 particle physics). The photoelectric effect and Compton scattering are similar effects, but are different in some very fundamental ways, as we will discover.

Study the Compton scattering diagram in figure (15.2). Let λ be the wavelength of the incident X-ray and λ' be the wavelength of the scattered X-ray. The momenta are thus

$$\begin{aligned} p &= \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}, \\ p' &= \cdots = \frac{h}{\lambda'}, \end{aligned} \tag{15.17}$$

since $c = \lambda\nu$, where ν is the frequency of the X-ray.

By conservation of momentum,

$$\mathbf{p} = \mathbf{p}' + \mathbf{p}_e, \tag{15.18}$$

where $\mathbf{p}$ is the momentum of the incoming X-ray, $\mathbf{p}'$ is the momentum of the scattered X-ray and $\mathbf{p}_e$ is the momentum of the electron. Squaring both sides gives

$$\begin{aligned} p_e^2 &= p^2 + p'^2 - 2\mathbf{p}\cdot\mathbf{p}' \\ &= p^2 + p'^2 - 2pp'\cos\vartheta, \end{aligned} \tag{15.19}$$

where ϑ is the scattering angle of the X-ray.

Before the collision, the electron has rest energy $E_0 = m_0c^2$. After the collision, the electron has energy $E = \sqrt{E_0 + p_e^2c^2}$. Conservation of energy thus implies

$$pc + E_0 = p'c + \sqrt{E_0 + p_e^2c^2}. \tag{15.20}$$

Collecting terms with c and squaring both sides yields

$$c^2(p - p') + E_0^2 + 2E_0c(p - p') = E_0^2 + p_e^2c^2. \tag{15.21}$$

Cancelling E_0^2 and dividing through by c^2 gives

$$p_e^2 = p^2 + p'^2 - 2pp' + \frac{2E_0(p - p')}{c}. \tag{15.22}$$

Combining equation (15.19) and equation (15.22) gives

$$\frac{E_0(p - p')}{c} = pp'\cos\vartheta. \tag{15.23}$$

Multiplying each term by $hc/pp'E_0$ and using $\lambda = h/p$ gives

$$\lambda - \lambda' = \frac{hc}{E_0}(1 - \cos\vartheta). \tag{15.24}$$

From here, using $E_0 = m_0c$ gives the usual Compton scattering equation for the difference in wavelength $\Delta\lambda \geq 0$ between the incoming photon and the scattered photon

$$\Delta\lambda = \lambda - \lambda' = \frac{h}{m_0c}(1 - \cos\vartheta). \tag{15.25}$$

Compton scattering is different to the photoelectric effect insofar that:

- the photon and electron are both scattered and that
- the electron is a free particle (its potential energy is zero everywhere) so has a continuum of energy levels.

These key differences help answer why a photon cannot transfer all of its energy to a free particle. Let's prove this statement more formally.

Let's assume that a photon can transfer all its energy to a free particle. This statement is made via energy conservation

$$h\nu + m_0c^2 = m_0c^2 + T, \tag{15.26}$$

where T is the kinetic energy of the electron after the collision with a photon. Cancelling the m_0c^2 terms in equation (15.26) yields

$$T = h\nu = \frac{hc}{\lambda} = p_e c. \tag{15.27}$$

The relativistic energy of the electron is

$$\begin{aligned} E_e &= \sqrt{(m_0c^2)^2 + (pc)^2} \\ \implies (m_0c^2 + T)^2 &= (m_0c^2)^2 + (pc)^2 \\ \implies (m_0c^2)^2 + T^2 + 2Tm_0c^2 &= (m_0c^2)^2 + (pc)^2 \\ \implies p_e^2c^2 + 2p_ecm_0c^2 &= p_e^2c^2 \\ \implies 2p_ecm_0 &= 0. \end{aligned} \tag{15.28}$$

Since c and m_0 cannot be equal to zero, this implies that $p_e = 0$, which implies that $\lambda \to \infty$. This is inconsistent so a photon cannot transfer all its energy to a free electron, as postulated in equation (15.26).