4

Classical Mechanics 4

4.1 Sketching graphs

All graphs must i) have axes, ii) have axis labels with units, iii) have an appropriate amount of tick marks with values specified and iv) demonstrate the correct physics. In general, a graph should not have i) a legend, ii) a title, iii) the equation of the curve shown or iv) gridlines. Information describing the graph (i–iii) should go in the caption.

4.2 Simple harmonic motion

Figure (4.1) shows a pendulum of mass m and length ℓ displaced by an angle θ from the vertical. The kinetic energy of the pendulum is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\ell\dot{\theta})^2 = \frac{1}{2}m\ell^2\dot{\theta}^2, \tag{4.1}$$

where $\dot{\theta}$ is the angular velocity of the pendulum. The vertical displacement of the mass from the point it is released to the bottom of the pendulum's motion (i.e., from $\theta = \theta_0$ to $\theta = 0$) is given by

$$y(\theta) = \ell - \ell\cos\theta_0 = \ell(1 - \cos\theta_0), \tag{4.2}$$

such that the potential energy of the pendulum is

$$V(y) = mgy = mg\ell(1 - \cos\theta_0). \tag{4.3}$$

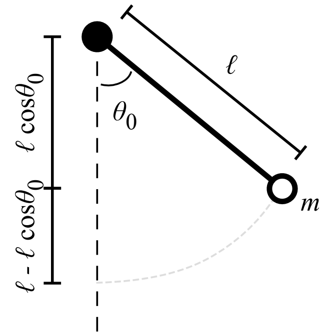

Figure 4.1: A pendulum of length ℓ and mass m with an arbitrary initial angle θ_0 with respect to the vertical (dashed line - -) is shown.

Given a pendulum with $T+U=$ const.,
where T is the kinetic energy and U is the potential energy,

a) the system is conservative	b) at $\theta_0 = 180°$, the pendulum never moves
c) the oscillations about $\theta = 0$ will get smaller	d) the motion is always simple harmonic

Assuming that the pendulum system contains only conservative forces (i.e., there are no dissipative forces such as friction), then the total mechanical energy

$$T + V = \text{const.} \tag{4.4}$$

This further implies that

$$\Delta T + \Delta U = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 + mgh_f - mgh_i = 0, \tag{4.5}$$

where v_f is the final velocity (i.e., the velocity at the point along the trajectory that we are interested in), v_i is the initial velocity (which is instantaneously zero when we release it from rest), $h_f = \ell - \ell\cos\theta$ is the final height of the pendulum at some angle θ and $h_i = \ell\cos\theta$ is the initial height of the pendulum from which it was released.

The potential energy is a maximum when the pendulum is at $\theta = \theta_0$ and a minimum at $\theta = 0$. The kinetic energy is a minimum at $\theta = \theta_0$ (it is zero) and a maximum at $\theta = 0$.

What is the velocity at any angle θ along the trajectory of the pendulum? Can we use a small angle approximation for an initial angle of $\theta_0 = 20°$? How good is this approximation?

We shall generalise equation (4.3) to consider the velocity at any point, not just at the bottom (where the velocity is a maximum).

The difference in height of the pendulum and its maximum position is always $\Delta y = y_f - y_i = \ell - \ell\cos\theta - (\ell - \ell\cos\theta_0) = \ell\cos\theta_0 - \ell\cos\theta$. From equation (4.5), and noting that $v_i = 0$, we have

$$\frac{1}{2}mv_f^2 + mg(\ell\cos\theta_0 - \ell\cos\theta) = 0 \tag{4.6}$$

which gives

$$v_f^2 \equiv v_f^2(\theta) = 2g\ell(\cos\theta - \cos\theta_0), \tag{4.7}$$

for all angles θ and θ_0.

We can further invoke the small angle approximation for $\cos\theta$ which states that

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \mathcal{O}(\theta^3). \tag{4.8}$$

Equation (4.7) is thus approximately

$$v_f^2 \approx 2g\ell\left(1 - \frac{\theta^2}{2} - 1 + \frac{\theta_0^2}{2}\right) = g\ell\left(\theta_0^2 - \theta^2\right). \tag{4.9}$$

How accurate is the small angle approximation for an angle of 20°? Let's look at the specific case of $\theta = 0$ where the pendulum is at its lowest point. Since we have used the small angle approximation, we must convert our angle to radians. Taking the ratio of the approximate velocity $\tilde{v}_f$ using equation (4.9) and the exact velocity using equation (4.7), we have

$$\frac{\tilde{v}_f^2(0)}{v_f^2(0)} \approx \frac{(20\frac{\pi}{180})^2}{2(\cos 0 - \cos 20\frac{\pi}{180})} \implies \frac{\tilde{v}}{v} \approx 1.005. \tag{4.10}$$

This represents a 0.5% difference between the exact velocity and approximate velocity using the small angle approximation.

4.3 (Optional): Equation of motion

In year 2 classical mechanics, you will learn that specifying the kinetic energy and potential energy of most systems is enough to derive the equation of motion. The equation of motion of a pendulum of arbitrary amplitude can be derived as

$$\ddot{\theta}(t) + \frac{g}{\ell}\sin\theta = 0. \tag{4.11}$$

This equation is valid for all initial angles of the pendulum. It is, however, very difficult to write down the exact solution of this differential equation (the exact solution is in terms of so-called Jacobi elliptic functions, which you do not learn at any point during your degree). We can, however, make some approximations. Assuming that θ is small (less than around 20°), we can invoke the small angle approximation

$$\sin\theta \approx \theta. \tag{4.12}$$

The equation of motion in equation (4.11) simplifies to

$$\ddot{\theta}(t) + \frac{g}{\ell}\theta = 0. \tag{4.13}$$

The error in the solution θ is around 1% if the initial angle $\theta(0)$ is less than around 20°. The error dramatically increases for larger amplitudes and the approximation no longer holds.

Equation (4.13) has exact solutions

$$\theta(t) = A\sin\big(\omega t + \varphi\big), \tag{4.14}$$

where A is the amplitude of the motion, $\omega = \sqrt{g/\ell}$ is the natural frequency of the pendulum and φ is a phase shift.