5

Oscillations

Consider an object moving under simple harmonic motion.
It's position as a function of time is $x(t) = A\cos(\omega t + \varphi)$.
Which statements are true?

a) $[A] = L$, $[\omega] = T^{-1}$ and φ is an angle.	b) A is the amplitude and ω is the frequency of oscillation.
c) φ shifts the origin of time	d) If $A = B\sin x$, B is the amplitude of oscillation.

5.1 Equilibria

Consider a particle subject to a conservative force F such that

$$U(x) = -U_0 \cos\left(\frac{\pi x}{\Lambda}\right), \tag{5.1}$$

where $U_0 > 0$ and $\Lambda = 0.1$ m. At which points will the particle be in a stable equilibrium if it had zero velocity?

Which functions are simple harmonic?

a) $x(t) = A\sin(\omega t)$	b) $x(t) = A\cos(\omega t) + B$
c) $x(t) = A\cos^2(\omega t)$	d) a train of Gaussians

A stable equilibrium implies that there is no net force acting on the particle, and that a small perturbation of its position results in a restorative force back to the original position. The first condition implies that

$$F_x(x_0) = -\frac{\mathrm{d}U(x_0)}{\mathrm{d}x} = 0. \tag{5.2}$$

At what positions x is the particle in a stable equilibrium? The particle is in a stable equilibrium when the potential energy is a minimum. The potential is a minimum when

$$\cos\left(\frac{\pi x}{\Lambda}\right) = 1. \tag{5.3}$$

This implies that

$$\frac{\pi x}{\Lambda} = 2\pi n, n \in \mathbb{Z}, \tag{5.4}$$

such that the points of stable equilibrium are integer (positive and negative) multiples of Λ,

$$x = 2\Lambda n. \tag{5.5}$$

The force acting on the particle is

$$F = -\frac{\mathrm{d}U(x)}{\mathrm{d}x} = -\frac{\pi}{\Lambda}U_0 \sin\left(\frac{\pi x}{\Lambda}\right). \tag{5.6}$$

When $|x| \ll \Lambda/\pi$, the small angle approximation can be used and the force reduces to something of the form

$$F \propto -x, \tag{5.7}$$

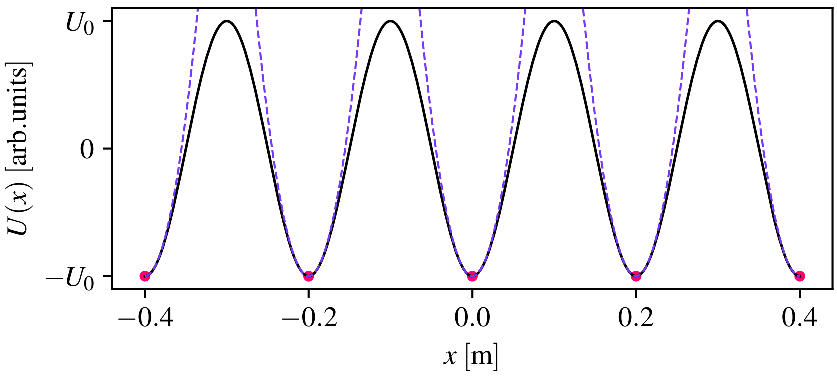

Figure 5.1: The sinusoidal potential (solid, black lines —) in equation (5.1) is shown as a function of position. The points of stable equilibria on the minima of the potential at $x = 0, \pm 2\Lambda, \pm 4\Lambda$ are shown (solid, pink circles). At the stable equilibria, approximate harmonic potentials are shown (dashed, purple lines - -). In the vicinity of the stable equilibria, the potential energy in equation (5.1) is well approximated by a harmonic potential.

which is a restorative force describing simple harmonic motion.

Figure (5.1) shows the potential energy function in equation (5.1) alongside the approximate harmonic potentials in the vicinity of the stable equilibria.

This is actually very profound: the harmonic oscillator (classical and quantum) is one of the most important models that you will study. What we have shown is that our periodic, sinusoidal potential can be well-approximated by a harmonic potential in the vicinity of a point of stable equilibria (about $|x| \ll \Lambda/\pi$).

Almost all potentials (especially in experimental physics) can be modelled as a harmonic potential in the vicinity of stable equilibria. We do this especially in quantum mechanics as it is one of few systems for which an analytical solution exists.

(Optional): Proof of stable/unstable equilibria

The second condition can be described mathematically. Let us consider an initially small, time-dependent perturbation $\xi(t)$ about an equilibrium point

x_0, such that the position of the particle is

$$x(t) = x_0 + \xi(t). \tag{5.8}$$

From Newton's second law,

$$F(x) = m\ddot{x}(t) = m\ddot{\xi}(t) = -\frac{\mathrm{d}U[x_0 + \xi(t)]}{\mathrm{d}x}. \tag{5.9}$$

Taylor expanding about x_0, in increasing powers of ξ, gives

$$m\ddot{\xi}(t) = -\frac{\mathrm{d}U(x_0)}{\mathrm{d}x} - \xi\frac{\mathrm{d}^2U(x_0)}{\mathrm{d}x^2} + \mathcal{O}(\xi^2). \tag{5.10}$$

Equation (5.2) tells us that the first term in the Taylor expansion is zero, so—to first order in ξ—we have

$$\ddot{\xi} + \frac{\xi}{m}\frac{\mathrm{d}^2U(x_0)}{\mathrm{d}x^2} = 0. \tag{5.11}$$

If $\frac{\mathrm{d}^2U(x_0)}{\mathrm{d}x^2} < 0$, the solutions to the differential equation are exponentials of the form

$$\xi(t) = A\mathrm{e}^{\alpha t} + B\mathrm{e}^{-\alpha t}, \tag{5.12}$$

where

$$\alpha = \sqrt{-\frac{1}{m}\frac{\mathrm{d}^2U(x_0)}{\mathrm{d}x^2}}. \tag{5.13}$$

As $t \to \infty$ (i.e., as the system evolves in time), ξ grows larger without bound and the solution never returns to the neighbourhood of the stable equilibrium x_0. This is referred to as an unstable equilibrium.

On the other hand, if $\frac{\mathrm{d}^2U(x_0)}{\mathrm{d}x^2} > 0$ (which is what we have), the solution $\xi(t)$ is simple harmonic, i.e.,

$$\xi(t) = A\sin(\omega t) + B\cos(\omega t), \tag{5.14}$$

where

$$\omega = \sqrt{\frac{1}{m}\frac{\mathrm{d}^2U(x_0)}{\mathrm{d}x^2}}. \tag{5.15}$$

As $t \to \infty$, ξ does not grow without bound and remains in a neighbourhood of x_0. This is referred to as a stable equilibrium.