6

Classical Mechanics 5

6.1 Types of orbit

All orbits are conic sections — the intersection of the surface of a cone with a plane. The conic sections are circles, ellipses, parabolae and hyperbolae.

The equation of a circle is

$$x^2 + y^2 = a^2, \tag{6.1}$$

where a is the length of the semi-major axis. The equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{6.2}$$

where b is the length of the semi-minor axis (if $a = b$, this reduces to the equation of a circle). The equation for a parabola is

$$y^2 = 4ax. \tag{6.3}$$

The equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{6.4}$$

noting the sign difference with an ellipse.

We also (heuristically) define the eccentricity ε of a conic section as the deviation of a conic section is from being circular — the eccentricity uniquely defines the shape of a conic section. For a circle, $\varepsilon = 0$; for an ellipse, $\varepsilon \in (0,1)$; for a parabola, $\varepsilon = 1$ and for a hyperbola, $\varepsilon > 1$. In the limiting case $\varepsilon \to \infty$, we get a line.

Figure (6.1) shows the four conic sections. Mathematically, the hyperbola features two symmetric curves, although only one is shown in the figure. We say that the orbiting body occupies one of the curves, and the other curve is just its mathematical image.

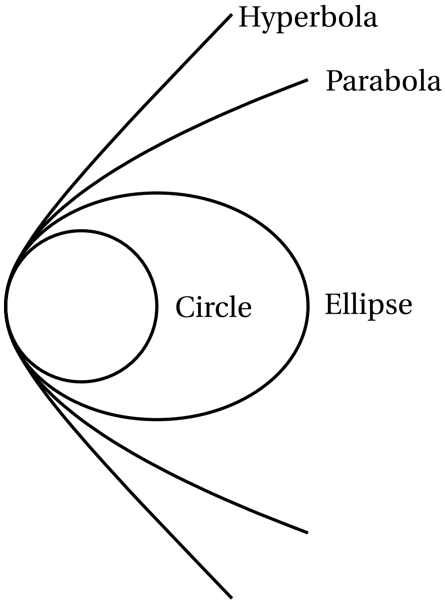

Figure 6.1: The four conic sections are shown. In a small region, the conic sections are approximately equal to each other. Outside of this region, they behave in very different ways.

We can also look at energy arguments of each trajectory by examining figure (6.1). We typically use the specific kinetic and potential energies in orbital mechanics, which are the typical equations divided by mass (so have units of energy per unit mass). In other words, the specific kinetic energy is

$$\mathscr{E}_T = \frac{1}{2}v^2, \tag{6.5}$$

and the specific (gravitational) potential energy is

$$\mathscr{E}_P = -\frac{GM}{r}. \tag{6.6}$$

Which conic section corresponds to an orbit with a total characteristic energy $\mathscr{E} = \mathscr{E}_T + \mathscr{E}_P > 0$?

a) Circle	b) Ellipse
c) Parabola	d) Hyperbola

The total specific energy is the sum of kinetic and potential energies, i.e.,

$$\mathscr{E} = \mathscr{E}_T + \mathscr{E}_P = \frac{1}{2}v^2 - \frac{GM}{r} = \text{const.} \tag{6.7}$$

Let us look at the three cases of the sign of $\mathscr{E}$.

1. **Case I:** $\mathscr{E} = 0 \implies \frac{1}{2}v^2 = \frac{GM}{r}$. We identify this as the minimum kinetic energy required to escape the gravitational potential of a massive body.

   Looking back at our diagram of the conic sections in figure (6.1), we identify this as the ellipse with eccentricity $\varepsilon = 1$.

   If we had any more energy, we would follow hyperbolic orbits. If we had less energy, we would follow elliptical orbits (or circular if $r = a$). Thus immediately we can predict the types of orbit for $\mathscr{E} > 0$ and $\mathscr{E} < 0$.

2. **Case II:** $\mathscr{E} > 0$ implies a hyperbolic orbit and

3. **Case III:** $\mathscr{E} < 0$ implies an elliptical orbit (with a circular orbit being a special case of this).

The total characteristic energy tells us whether we have an excess or too little energy to escape from a massive body.

Sam Wann goes on a tour of the Solar System. He starts with the Moon, where he has fun experiencing the low gravity where $g_M \approx 1.6\,\mathrm{m\,s^{-2}}$. He crouches as low as his spacesuit allows, which lowers his centre of gravity by 20 cm, jumps vertically, and reaches a height of 1.2 m above the ground.

He is looking forward to do the same when he will be on a remarkable asteroid he is planning to visit. The asteroid has a radius of 2.0 km and is perfectly spherical, where $g_A = 1.1 \times 10^{-3}\,\mathrm{m\,s^{-2}}$. If he were to do this, would he be at risk of reaching the escape speed at the surface of this asteroid?

The escape speed is the minimum energy Sam Wann requires to escape the orbit of the asteroid of mass M_A and radius R_A, i.e., the specific total energy

$$\frac{1}{2}v^2_{\text{escape,A}} - \frac{GM_A}{R_A} = 0 \implies v^2_{\text{escape,A}} = \frac{2GM_A}{R_A}. \tag{6.8}$$

The mass of the asteroid is given by the definition of the acceleration due to gravity

$$g_A = \frac{GM_A}{R_A} \implies M_A = \frac{g_A R_A^2}{G}. \tag{6.9}$$

The escape speed in terms of the acceleration due to gravity on the asteroid and the radius of the asteroid is therefore

$$v^2_{\text{escape,A}} = 2g_A R_A. \tag{6.10}$$

Given $g_A = 1.1 \times 10^{-3}\,\mathrm{m\,s^{-2}}$ and $R_A = 2.0 \times 10^3$, $v_{\text{escape}} = 2.1\,\mathrm{m\,s^{-1}}$.

We then must compare this speed to the maximum speed Sam obtains when he jumps on this asteroid. The motion of the jump is described by two forces: the upwards force F imparted by Sam's legs on the ground, and the force due to his weight mg_A which acts in the opposite direction. A free-body diagram shows these forces clearly.

The vertical acceleration on the asteroid is therefore

$$a_{y,A} = \frac{F}{m} - g_A. \tag{6.11}$$

Assuming the force F to be constant throughout the jump, we can describe Sam's jump speed using

$$v_{\text{jump,A}}^2 = 2\left(\frac{F}{m} - g_A\right)s, \tag{6.12}$$

where $s = 20$ cm is the displacement of his centre of mass.

What is F/m? We can find an equivalent expression on the moon, assuming that his mass doesn't change and that the force he exerted by his legs is the same. Therefore,

$$v_{\text{squat,M}}^2 = 2\left(\frac{F}{m} - g_M\right)s. \tag{6.13}$$

Sam extends from his deep squat, with some final velocity $v_{\text{squat,M}}^2$, and goes immediately into a jump of height h. The initial velocity of the jump is $v_{\text{squat,M}}^2$, and the final velocity is 0. This second part of the dynamics is described by another equation of constant acceleration,

$$0 = v_{\text{squat,M}}^2 - 2g_M h. \tag{6.14}$$

Combining equations (6.13) and (6.14) gives an expression for F/m in quantities which we know

$$\frac{F}{m} = \left(1 + \frac{h}{s}\right)g_M. \tag{6.15}$$

We can now determine the speed of the jump on the asteroid using equation (6.12) as

$$\begin{aligned} v_{\text{jump,A}}^2 &= 2\left[\left(1 + \frac{h}{s}\right)g_M - g_A\right]s \\ &= 2\left[\left(1 + \frac{1.2}{0.2}\right)1.6 - 1.1\times10^{-3}\right]\cdot 0.2 \\ \implies v_{\text{jump,A}} &= 2.1\ \text{m s}^{-1}. \end{aligned} \tag{6.16}$$

The escape speed $v_{\text{escape,A}} = 2.1\ \text{m s}^{-1}$ and Sam's jumping speed $v_{\text{jump,A}} = 2.1\ \text{m s}^{-1}$ are the same to 2 significant figures. Sam better be careful.

Let us consider two points of Sam's body, one at a distance r of the centre of Jupiter and one at at distance $r+\delta r$. Assuming that these two points have the same mass, the magnitude of their gravitational acceleration is, respectively, a and $a-\delta a$. Show that

$$\delta a \approx 2Gm_J \frac{\delta r}{r^3},$$

where m_J is the mass of Jupiter, and find an order of magnitude estimation for δa.

δa is the difference in the acceleration due to gravity between two points on Sam's body. It is exactly given as

$$\delta a = \frac{Gm_J}{r^2} - \frac{Gm_J}{(r+\delta r)^2}. \tag{6.17}$$

We can pull out a common factor of $\frac{1}{r^2}$ if we factorise the second fraction as

$$\begin{aligned} (r+\delta r)^2 &= r^2 + \delta r^2 + 2\delta r \cdot r \\ &= r^2\left(1 + \frac{\delta r^2}{r^2} + \frac{2\delta r}{r}\right) \\ &= \left(1 + \frac{\delta r}{r}\right)^2. \end{aligned} \tag{6.18}$$

Thus,

$$\delta a = \frac{Gm_J}{r^2}\left[1 + \frac{1}{\left(1+\frac{\delta r}{r}\right)^2}\right]. \tag{6.19}$$

A health warning arises here: r will be a very large number. δr will, by definition, be very small as $\delta r/r \ll 1$. If you attempt to put values of these into a standard calculator, you will most likely run into memory and rounding issues. [This issue also arises in special relativity with the ratio $v/c \ll 1$]. We must therefore use a binomial approximation to remove such issues.

We note that $(1+\delta r/r)^2 \approx 1+2\delta r/r$ for small $|\delta r/r|$. Equation (6.19) reduces to

$$\delta a \approx \frac{Gm_J}{r^2}\left[1+\frac{1}{1+\frac{2\delta r}{r}}\right], \tag{6.20}$$

and simplifies easily to

$$\delta a \approx 2Gm_J\frac{\delta r}{r^3}, \tag{6.21}$$

as required.

What is an order of magnitude estimation for δa? Let $\delta r \sim 1$ m, $m_J \sim 10^{27}$ kg, $r = r_J \sim 10^8$ m and $G \sim 10^{-11}$ m^3 kg^{-1} s^{-2}. Then,

$$\delta a \sim \frac{10^{-11}\cdot 10^{27}\cdot 10^{0}}{(10^8)^3} = 10^{-7}\ \mathrm{m\,s^{-2}}. \tag{6.22}$$

This change in acceleration from the top and bottom of Sam is insufficient to wake him upon his approach to Jupiter.

[However, this acceleration is detectable by precision interferometers used to measure g on Earth. It is of a similar order of magnitude of the gravitational attraction between the Moon, a point on the Earth's surface, and the Earth's centre of gravity. This difference is responsible for tidal forces.

Whilst insignificant to us, acceleration changes of this order are important for other physical effects.]

6.2 Approximations in physics

Many problems in physics are analytically intractable — it is either impossible or very difficult to proceed without a computer. In order to make progress with some calculations in physics, we make use of approximations. There are two main approximations used in physics: i) the binomial approximation and ii) the Taylor or Maclaurin series expansion.

You will often not be told if a solution requires an approximation to proceed. You will, however, gain an intuition about when to use one.

Binomial approximation

You will sometimes meet the Binomial approximation in special relativity and in orbital mechanics. The binomial expansion states that

$$(1+x)^{\alpha} \approx 1+\alpha x \tag{6.23}$$

if $|x|<1$ and $|\alpha x| \ll 1$. Figure (6.2) shows the polynomial $(1+x)^{\frac{1}{2}}$ and its binomial approximation $1+\frac{1}{2}x$ over the domain $x \in [0,10]$. It can be seen that the approximation fails for $|x|>1$ with increasing error.

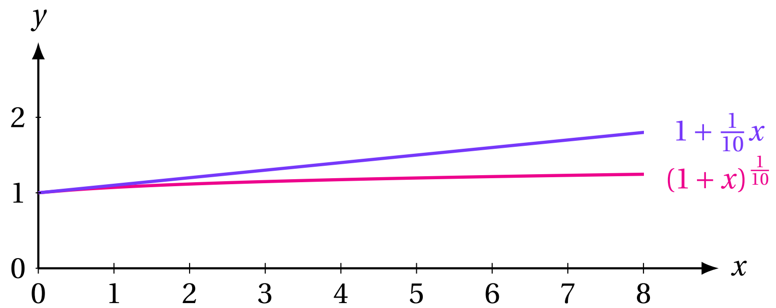

Figure 6.2: The polynomial $(1+x)^{\frac{1}{10}}$ and its binomial approximation $1+\frac{1}{10}x$ are shown over a range $x \in [0,8]$. The two curves match well for small values of x.

Proof

The function

$$f(x) = (1+x)^{\alpha}, \tag{6.24}$$

where $x, \alpha \in \mathbb{C}$, can be expressed as a Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x! = f(0)x^0 + f'(0)x^1 + f''(0)x^2 + f'''(0)x^3 + \cdots . \tag{6.25}$$

Then,

$$(1+x)^\alpha = 1 + \alpha x + \frac{1}{2}\alpha(\alpha-1)x^2 + \frac{1}{6}\alpha(\alpha-1)(\alpha-2)x^2 + \cdots . \tag{6.26}$$

If and only if $|x| < 1$ and $|\alpha x| \ll 1$, the series can be truncated so

$$(1+x)^\alpha = 1 + \alpha x + \mathcal{O}(x^2). \tag{6.27}$$

Sometimes, expressions which satisfy $|x| < 1$ and $|\alpha x| \ll 1$ reduce to zero if we only retain the first two terms of the series expansion. In these cases, we need to move to terms quadratic in x to avoid this trivial solution.

You typically meet binomial approximations in planetary motion, astrophysics, cosmology and special relativity.

Notation

When dealing with approximations, the equation $(1+x)^\alpha \approx 1 + \alpha x$ is correct (with an approximation sign). The equation $(1+x)^\alpha = 1 + \alpha x + \mathcal{O}(x^2)$ is exact. We use $\mathcal{O}(\cdot)$ to mean 'terms of the order of blah' in this context, so by definition it includes all terms in its expansion. You can also use $+\cdots$ rather than $\mathcal{O}(\cdot)$ to indicate more terms in the expansion, and this also is a statement of equality.

Small angle approximation

Recall the Maclaurin series for $\sin\theta$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \mathcal{O}(\theta^7), \tag{6.28}$$

where θ is strictly measured in radians [1]. Then, if θ is small,

$$\sin\theta \approx \theta. \tag{6.29}$$

Typically, if $\theta \lessapprox 20°$, then the approximation is correct to within 1%.

[1] Indeed, in calculus, *all* trigonometric functions require use of radians

Similarly,

$$\cos\theta \approx 1 - \frac{\theta^2}{2!}, \tag{6.30}$$

and

$$\tan\theta \approx \theta. \tag{6.31}$$

You typically meet small angle approximations in simple harmonic motion.