7

Waves and Optics 1

7.1 Standing and travelling waves

A standing wave is one where the peak amplitude does not move in time (but is free to oscillate in time about an equilibrium position). A travelling wave is one where the peak amplitude moves in time. Two travelling waves of the same angular frequency, but travelling in different directions, form a standing wave when linearly superimposed.

A travelling wave has a displacement of the form

$$y(t,x) = A\cos(kx - \omega t), \tag{7.1}$$

where A is the amplitude of oscillation, k is the wave number (or wave vector in multiple dimensions) and ω is the angular frequency of the wave. Since cos is periodic with period 2π, we can derive the wavelength and period of this wave. If we move one wavelength λ along the x axis, we get the same value for the displacement back (this is the definition of periodicity). It is therefore true that

$$\begin{aligned} y(t,x) &= A\cos(kx - \omega t) \\ y(t,x+\lambda) &= A\cos(k(x+\lambda) + \omega t). \end{aligned} \tag{7.2}$$

When $y(t,x) = y(t,x+\lambda)$, we require that

$$k(x+\lambda) - \omega t = kx - \omega t + 2\pi, \tag{7.3}$$

thus

$$k\lambda = 2\pi. \tag{7.4}$$

The same argument holds when considering the period $t+T$ to find $\omega T = 2\pi$.

Imaging now an observer running along the x-axis with the travelling wave. The speed of the wave is the speed the observer needs to travel at to observe no wave motion in their frame of reference. The observer's position is thus $x = vt$. This is a Gallilean transformation! Since $x \to x + vt$ to find the displacement that the observer measures

$$y(t,x) = A\cos(k(x+vt) - \omega t) = A\cos(kx - t(\omega - kv)). \tag{7.5}$$

If the displacement of the wave does not vary as the observer moves, then there can be no time-dependence. This must mean that $\omega - kv = 0$ to eliminate t. Then,

$$v = \frac{\omega}{k} = f\lambda. \tag{7.6}$$

7.2 Modelling a vibrating string

NB: You do not need to be able to do this derivation. In the tutorial, I demonstrated that it is often useful to decompose functions into a series of sine and cosine terms, which evolve like standing waves in time. You will learn the formalism of Fourier series in your mathematics courses.

Boundary conditions

When I am doing computational physics, I almost always care about the initial or boundary conditions of my problem (e.g., where in space it starts, with what velocity, whether the boundary is periodic, etc.).p

Let's consider a finite vibrating string which satisfies the wave equation

$$u_{tt} = c^2 u_{xx}, \tag{7.7}$$

where $x \in (0,L)$ and $t > 0$. The solution to this partial differential equation (PDE) is $u(t,x)$, i.e., a function which varies spatially and temporally. I may know information about this function at $t = 0$ (i.e., before it starts evolving as per the PDE)

$$u(0,x) = f(x). \tag{7.8}$$

I may also know information about the initial velocity at which the string is plucked (or otherwise displaced)

$$u_t(0,x) = g(x). \tag{7.9}$$

Equations (7.8) and (7.9) tell us about the initial shape and instantaneous velocity of the string.

We might want to fix one or two points on the string for all time, or to specify a force acting on those endpoints. We can use boundary conditions of the form

$$u(t,0) = h_1(t), \quad u(t,L) = h_2(t), \quad \forall\, t, \tag{7.10}$$

where, in this case, h_1 and h_2 are (possibly time-dependent) amplitudes of the string at the endpoints.

Which function gives rise to $u(t,0) = u(t,L) = 0\ \forall\, t$?

a) $\sim \sin x$	b) $\sim \cos x$
c) $\sim \sin x \cos x$	d) $\sim x$

Fourier series

Fourier series are basically the wave and circular equivalent of Taylor series for polynomials. They are a way to represent periodic functions [1] as a sum of sin and/or cos waves. For a function $f(x)$ with period $2L$,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right], \tag{7.11}$$

[1] You can sometimes use Fourier transforms on problems which are non-periodic, although you will then introduce periodicity into the problem.

where

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \tag{7.12}$$

and

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{7.13}$$

The term $\frac{1}{2}a_0$ is the average value (sometimes referred to as the 0th harmonic) of $f(x)$ over its period, i.e., $\langle f(x)\rangle = \frac{1}{2L}\int_0^{2L} f(x)\mathrm{d}x$. The infinite series in equation (7.11) then builds upon this average (or, 'initial guess' of the form of $f(x)$) by adding successive sin and cos functions to find the true approximation (or equality if the sum is infinite) for $f(x)$. Remember that these equations are for a period of $T = 2L$ — if you wanted to half the period then you must substitute $T/2 = L$.

Modelling the plucked string

Let's say we pluck a guitar string of length L by a height h at some position $x = d$ along the body of a guitar, as shown in figure (7.1). We wish to find an equation for the time evolution of the triangular wave, in terms of sine and cosine functions. In other words, we wish to decompose the triangular string into a number of individual modes, which add to give the overall shape of the string. The guitar string is fixed to both ends of the guitar, leading to a boundary condition of the form $\sin(n\pi/L)$, where n are the number of antinodes.

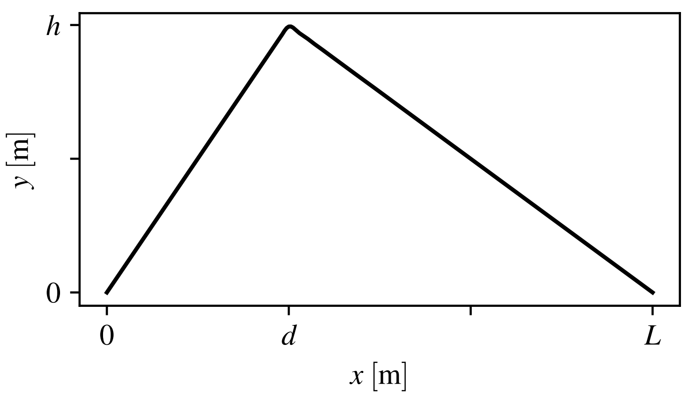

Figure 7.1: The expected triangular shape of the guitar string of length L, plucked a height h at a distance d along the string.

We can construct the transverse displacement of the string using the Fourier series

$$y(t,x) = \sum_{n=1}^{N} [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)] \sin\left(\frac{n\pi x}{L}\right), \tag{7.14}$$

where we have decomposed the function into sine and cosine waves as per equation (7.11) and applied the boundary conditions for each mode that we are decomposing the string into. We identify the frequency of each mode as

$$\omega_n = 2\pi \frac{nv}{2L}, \tag{7.15}$$

where $v = \sqrt{T/\mu}$ is the speed of the string with tension T and linear density μ. We identify ω_n as the natural frequencies of each mode. The Fourier coefficients A_n and B_n are associated with the initial displacement and initial velocity, respectively, and are given by

$$A_n = \frac{2}{L} \int_0^L y(0,x) \sin\left(\frac{n\pi}{L}\right) \mathrm{d}x, \tag{7.16}$$

and

$$B_n = \frac{2}{\omega_n L} \int_0^L u(0,x) \sin\left(\frac{n\pi}{L}\right) \mathrm{d}x. \tag{7.17}$$

Given that the initial velocity of the string is 0, we immediately have that $B_n = 0 \forall n$. Given also that A_n is simply the integral over two straight lines from figure (7.1), one has that

$$A_n = \frac{2h}{n^2\pi^2} \frac{L^2}{d(L-d)} \sin\left(\frac{n\pi d}{L}\right). \tag{7.18}$$

What happens as $n \to \infty$? We observe that the frequency ω_n increases and that each successive mode we add gets smaller in amplitude—eventually we can truncate our Fourier series (much like we can truncate a Taylor series).

The Fourier series expansion of the string at $t = 0$ in equation (7.14) is thus

$$y(t=0,x) = \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi x}{L}\right). \tag{7.19}$$

This initial wave function then evolves via the wave equation.

Figure (7.2) shows some examples of what terms in the summation in equation (7.19) look like. The summation of these modes produces an approxi-

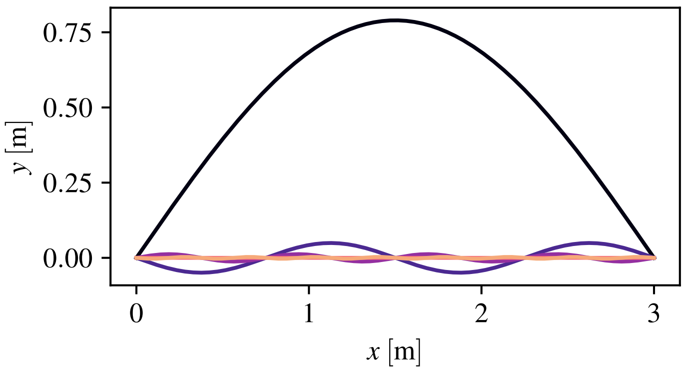

Figure 7.2: 4 modes in the summation in equation (7.19) are shown. The amplitude of each mode decreases as n increases.

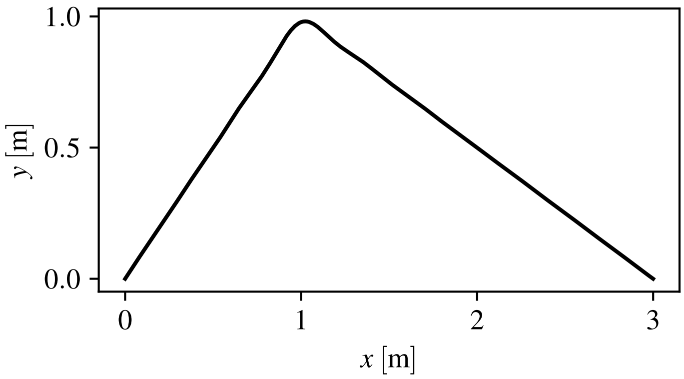

Figure 7.3: 4 modes in the summation in equation (7.19) are shown. The amplitude of each mode decreases as n increases. Even at $n = 4$, the triangular shape we require starts to appear. This blows my mind!

mation for the shape of our guitar string, entirely in terms of sine and cosine functions. The approximation is shown in figure (7.3).

And that is it! We have decomposed the guitar string into a series of sine and cosine waves, adding modes of successively smaller amplitudes, which form a triangular shape. Our standing waves can be linearly superimposed to form a triangular wave, which we can then propagate forward in time.